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On the transition to planing of a boat

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This paper is concerned with the problem of the transition to planing of a boat. A steady-state nonlinear solution of the problem is obtained using a theory of fluid sheets for two-dimensional motion of an incompressible inviscid fluid contained in the paper of Green & Naghdi (1977), together with appropriate jump conditions demanded by the theory. The motion of the fluid is coupled with the motion of a free-floating body and detailed analysis is undertaken pertaining to such features as the sinkage, the bow-up trim angle and the determination of the propulsion force. In particular, the results show that the governing mechanism of the hump speed phenomenon is the change in the bow-up trim angle of the boat. The differential equations and the relevant boundary conditions of the problem are reduced to a system of essentially algebraic equations whose solutions are obtained numerically.

1. Introduction

This paper is concerned with the nonlinear steady-state solution of the problem of the transition to planing of a two-dimensional boat; the term 'two-dimensional' boat is used here to indicate a boat whose beam (or width) is very large. Although the problem as treated here is concerned with the motion of a boat moving with a constant horizontal velocity over a fluid whose equilibrium depth is constant, it is mathematically more convenient to consider the equivalent problem of the steady two-dimensional motion of an incompressible inviscid fluid in the absence of surface tension past a stationary free-floating body on an otherwise free top surface of the fluid. Even when attention is confined to such steady two-dimensional motions of an inviscid incompressible fluid over a level bottom, the difficulties associated with obtaining exact analytical solutions of the (nonlinear) three-dimensional equations of motion and appropriate boundary conditions are so far insurmountable. It is therefore useful to consider alternative approaches. The most common alternative is to simplify the three-dimensional equations of motion by a systematic linearization procedure. Another possibility is to derive for a given class of problems alternative nonlinear equations of motion by asymptotic techniques (see e.g. Peregrine 1972). Still another alternative is to construct an appropriate theory by direct approach based on the concept of a directed fluid sheet (Green & Naghdi 1976*a*). In this paper we pursue the latter alternative and utilize the differential equations of motion of the restricted theory of a directed fluid sheet in the form derived by Green & Naghdi (1977).

Although the present formulation of the problem of steady two-dimensional flow past a stationary obstacle is sufficiently general to include a study of the problem of the squat of a ship or the flow past a sluice gate of a general shape, in this paper we confine attention to the problem of transition to planing with emphasis on the inter-

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relationship between the maximum bow-up trim angle and the so-called hump speed. [This terminology usually refers to the speed at which a boat experiences maximum resistance to motion. Additional remarks concerning a suitable definition of the hump speed are made later in this section and in §5.]

Before proceeding further, it is desirable to indicate briefly the manner in which the analysis of the problem of a free-floating boat is carried out in the present paper. Basically, we begin by specifying the orientation of the boat or ship (relative to the fixed bottom of the fluid), use the consequence of the linear momentum principle to determine the fluid flow past the obstacle, and then analytically obtain the pressure distribution on the fluid–boat interface. After calculating the drag, lift and moment exerted by the fluid on the boat, the boat's orientation is adjusted until the appropriate equations of linear momentum and angular momentum associated with the motion of the boat are satisfied. To elaborate, we first 'guess' the orientation of the boat and calculate the forces and moment (about the boat's centre of mass) which are exerted by the fluid on the boat. Then, by iteration, we continue to correct the initial 'guess' until these forces and moment exactly balance those due to the boat's weight and propulsion system. Details of the numerical procedure used in the paper are discussed in §5.

In the rest of this section we elaborate on the contents of the paper, along with relevant background pertaining to various aspects of the planing problem. The main kinematics and the differential equations governing the two-dimensional motion of an incompressible, homogeneous, inviscid fluid sheet are summarized in §2. These equations, which include the effects of gravity and vertical inertia, are those of a restricted theory of a directed fluid sheet (Green & Naghdi 1977) and are specialized here for propagation of fairly long waves over water of constant initial depth. Next, in §3, attention is called to the fact that the steady-state equations which describe the shape of the free surface may be integrated in terms of an elliptic integral of the first kind. The steady-state equations can also be solved analytically when the location of the top surface is specified, rather than being free and unknown. Evidently, the problem of fluid flow past an obstacle sketched in figure 1† may best be analysed by considering the solutions in separate regions of flow, labelled as regions I, II, III in figure 1, and by matching these solutions with the aid of jump conditions demanded by the theory and the particular problem under consideration. These jump conditions are recorded in an appendix at the end of the paper. Boundary conditions are also imposed which reflect the assumption that far ahead of the body the fluid flows as a uniform stream.

The equations of motion of a free-floating boat (regarded here as a rigid body) are discussed in §4 and provide the necessary coupling between the fluid motion and the motion of the boat. For the fairly general hull shapes considered in the present paper, the equations describing the orientation of the free-floating body involve certain definite integrals associated with the forces and moment that the fluid exerts on the body, but these equations are essentially algebraic. Thus, in §4, the analysis of the flow past a free-floating body is reduced to solving a system of essentially algebraic equations.

The motion of a planing craft – as speed is increased from zero – is usually charac-

† Our designation of x_3 and x_4 (rather than x_2 and x_3) on the abscissae in figure 1 is simply for later notational consistency.

terized by an increase in the bow-up trim angle to a maximum, followed by a decrease in trim angle. Here we define the hump speed as the speed at which the bow-up trim angle reaches its maximum value and consider a planing speed to be any speed greater than the hump speed. [Most authors define the hump speed as the speed for which resistance to motion is a maximum. The difference between this and our definition is discussed in §5.] For the purpose of this paper, we need to recall only the works of Squire (1957) and Cumberbatch (1958), but call attention to a more complete account of the planing problem by Wehausen & Laitone (1960, pp. 587–592).

Both Squire (1957) and Cumberbatch (1958) utilize the linearized three-dimensional equations of an incompressible inviscid fluid for two-dimensional irrotational steady motions. They consider planing over water of infinite depth and use certain results from Lamb (1932, §§242–244) and develop an integral equation which relates the pressure distribution on the top surface of the fluid to the shape of the top surface. Then, after approximating the pressure distribution on the hull, the associated hull shape may be calculated by using the integral referred to above. In order for this associated hull shape to closely resemble the desired hull shape, other approximate (and perhaps more complex) pressure distributions must be considered. While such a procedure may be used to approximate more general hull shapes, it can be accomplished only at the expense of increased analytical complexity. More specifically, Squire (1957) considers the planing of a wedge over water of infinite depth, and he restricts the motion of the wedge in such a manner that the bow-up trim angle is always constant. He further assumes that for all speeds above zero the location of the trailing edge is so specified that the fluid separates smoothly from the bottom corner of the wedge and the fluid pressure is atmospheric there. In order to explain the hump speed phenomenon, Squire (1957) introduces a frictional force. His analysis appears to suggest (see figure 9 in Squire's paper) that, at least for crafts which plane with a constant trim angle, the hump speed phenomenon is entirely controlled by frictional effects. A further contribution to the subject by Cumberbatch (1958) deals with the problem of transition to planing at high Froude numbers of crafts with hull geometries which are plane, parabolic or some combination of the two.

The developments of §§3–4 are readily applicable to the problem of transition to planing of a self-propelled free-floating boat of general hull shape over water of finite depth† with a level bottom. Since our steady-state formulation of the problem (§§3–4) is given in the context of the nonlinear theory, the determination of the location of the trailing edge is part of the solution: it is not specified as in the papers of Squire (1957) and Cumberbatch (1958), but rather is determined by the condition that the fluid pressure be atmospheric there. An example of the transition to planing of a boat-like body is solved in §5. With the aid of a computer program, the definite integrals associated with the forces and moment that the fluid exerts on the boat are evaluated by a Gaussian quadrature algorithm and the essentially algebraic equations describing the orientation of the free-floating boat are solved by a Newton–Raphson iteration procedure. The complete solution of the problem involves also the solution for the

† The term 'finite depth' is used here to distinguish between the analysis of the present paper and those of others cited earlier in this section which are valid for infinite depth. We emphasize that the theory of a directed fluid sheet employed here is not applicable to water of infinite depth and that the system of differential equations (2.6)–(2.7) is necessarily approximate compared to the (exact) system of three-dimensional equations of motion.

shape of the free surface in the trailing region characterized by a nonlinear ordinary differential equation whose solution is in the form of a cnoidal wave. While a solution of this kind is analytically well understood, it is more convenient to solve the differential equation numerically by a Runge–Kutta integration scheme. Various aspects of this solution are exhibited in figures 3–5. These include the orientation of the boat in different stages of planing (figure 3) and the locations of the leading and trailing edges (figure 5). The results also show (see figure 4(*b*, *c*)) that the hump speed can be predicted by a purely inviscid theory and that the controlling mechanism is the change in the bow-up trim angle.

It is well known that an inviscid theory is incapable of completely describing the drag experienced by a boat or ship. Nevertheless, it is generally assumed that an inviscid theory can accurately predict the orientation of a boat or ship relative to the fixed bottom. At the end of §5, we briefly discuss the results of a certain calculation for the transition to planing of a boat in which account is taken of frictional effects through the use of the coefficient of friction given by the I.T.T.C. 1957 model–ship correlation line. These results of this numerical example suggest that the sinkage and trim predictions of the purely inviscid theory are quite accurate.

Our general developments (§§3–4), as well as the specific example of the problems of planing in §5, are concerned with the fluid flow past a free-floating body whose hull geometry has continuous curvature. These results are not immediately applicable to a hull shape which does not have continuous curvature, such as the wedge considered by Squire (1957). In order to study the transition to planing of a wedge, we need to modify slightly the formulation contained in §§3–4. Thus, the necessary modifications are discussed in §6 and examples of the transition to planing of a wedge and a wedge-like boat are solved in §7. [It should be noted that it is necessary to distinguish here between a wedge which has discontinuous slope at its bottom corner, and wedge-like boat which has continuous curvature.] It is shown (figure 6) that except for near-zero speed and the hump speed, the idealizations associated with the planing of a wedge are quite useful in accurately predicting the sinkage and trim of the wedge-like boat.

Before closing this section, it is desirable to indicate why the theory of a directed fluid sheet (Green & Naghdi 1976*a*, 1977) can be applied to the problem of transition to planing of a boat. The (three-dimensional) problem under discussion involves a boat in contact with a body of water of finite depth; the former is regarded as a rigid body and the latter is modelled here by a directed fluid sheet. In this connexion, we first note that the system of differential equations employed accounts for the effect of vertical inertia, is translation invariant and satisfies exactly the boundary conditions on the top and bottom surfaces of the fluid sheet. Moreover, the theory is sufficiently general to allow: (1) the prescription of appropriate jump conditions for modelling the fluid behaviour at the attachment point of the leading edge of the boat; (2) the determination of the pressure on the bottom surface of the boat (due to the fluid motion); and (3) the determination of the detachment point in the trailing edge of the boat.

Clearly the transition problem is nonlinear in nature; and the source of the nonlinearities, which manifest themselves in the solution, are mainly those arising from (*a*) large changes in the orientation of the boat including the trim angle, (*b*) the manner in which the resultant forces and moment at the leading edge of the boat are accounted for, (*c*) the treatment of the detachment point in the trailing region and (*d*) the shape of the free surface in the trailing region. Although the theory of a directed fluid sheet used is an approximate nonlinear theory relative to the exact three-dimensional theory, it is nevertheless capable of allowing for all of the above nonlinearities and the present solution may at least be regarded as providing limited information about a difficult problem.

2. Basic equations

We record here the nonlinear differential equations governing the two-dimensional motion of an incompressible, homogeneous, inviscid fluid sheet. The two-dimensional motion is confined to the x, z plane of a fixed system of rectangular Cartesian co-ordinates (x, y, z) in which the velocity component in the y direction is zero, and the differential equations are appropriate for propagation of fairly long water waves. These equations, which include the effects of gravity and vertical inertia, follow by specialization from more general results of Green & Naghdi (1977) derived in the context of a restricted theory of a directed fluid sheet. Additional background material on recent developments in the direct formulation of the theory of fluid sheets, based on a continuum model called a *Cosserat* (or a *directed*) surface, may be found in a recent paper by Naghdi (1979).

We recall that a directed surface $\mathscr{C}$ comprises a material surface and a director assigned to every point of the material surface. Let the particles of the material surface of $\mathscr{C}$ be identified with a system of convected co-ordinates θ^α $(\alpha = 1, 2)$ and let the surface occupied by the material surface in the present configuration of $\mathscr{C}$ at time t be referred to as $\mathscr{s}$. Let $\mathbf{r}$ and $\mathbf{d}$ denote the position vector of a typical point of $\mathscr{s}$ and the director at the same point, respectively. Then, a motion of the directed surface $\mathscr{C}$ is specified by

$$\mathbf{r} = \mathbf{r}(\theta^\alpha, t), \quad \mathbf{d} = \mathbf{d}(\theta^\alpha, t) \tag{2.1a, b}$$

and the velocity and the director velocity are defined by

$$\mathbf{v} = \dot{\mathbf{r}}, \quad \mathbf{w} = \dot{\mathbf{d}}, \tag{2.2a, b}$$

where a superposed dot denotes the material time derivative holding θ^α fixed. With reference to the rectangular Cartesian co-ordinates introduced earlier and in the context of the restricted theory of a directed fluid sheet mentioned above, the position vector $\mathbf{r}$ and the director $\mathbf{d}$ can be represented in the form

$$\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 + \psi\mathbf{e}_3, \quad \mathbf{d} = \phi\mathbf{e}_3 \quad (\phi > 0), \tag{2.3a, b}$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are right-hand orthonormal base vectors along the Cartesian co-ordinate axes and where x, y, ψ, ϕ are functions of θ^α, t. The specification of $\mathbf{d}$ in the form (2.3b) can always be made in one configuration even in the context of a more general theory of a directed fluid sheet, but the director will not necessarily remain parallel to $\mathbf{e}_3$ throughout the motion. However, the theory discussed here (Green & Naghdi 1977) restricts the director to remain parallel to a fixed direction for all time.

For two-dimensional fluid motions confined to the x, z plane, we may identify y with the convected co-ordinate θ^2 and consider the motions in the x, z plane specified by $\theta^2 = 0$. Thus, in place of (2.3a, b), we may write

$$\mathbf{r} = x(\theta^1, t)\,\mathbf{e}_1 + \psi(\theta^1, t)\,\mathbf{e}_3, \quad \mathbf{d} = \phi(\theta^1, t)\,\mathbf{e}_3, \tag{2.4a, b}$$

where x, ψ, ϕ are now different from the corresponding functions in (2.3). Also, the velocity and the director velocity for two-dimensional motions in the x, z plane can be written as

$$\mathbf{v} = u\mathbf{e}_1 + \lambda\mathbf{e}_3, \quad \mathbf{w} = w\mathbf{e}_3, \tag{2.5a, b}$$

$$u = \dot{x}, \quad \lambda = \dot{\psi}, \quad w = \dot{\phi}. \tag{2.5c, d}$$

Then, the condition of incompressibility and the equations of motion over a level bottom are given by the following two sets of nonlinear partial differential equations (Green & Naghdi 1977):

$$w + \phi u_x = 0, \tag{2.6a}$$

$$\rho^*\phi\dot{u} = \hat{p}\beta_x - p_x, \quad \rho^*\phi\dot{\lambda} = -\rho^* g\phi + \bar{p} - \hat{p}, \quad \tfrac{1}{12}\rho^*\phi\dot{w} = -\tfrac{1}{2}(\hat{p}+\bar{p}) + \frac{p}{\phi} \tag{2.6b, c, d}$$

and

$$-\tfrac{1}{2}p\beta_x - s + S_x = 0, \quad \tfrac{1}{2}\phi p\beta_x - p\psi_x - (\phi S)_x = 0, \tag{2.7a, b}$$

where the subscripts denote partial differentiation with respect to x and the scalars S and s given by (A 3) of the appendix are arbitrary functions of x, t. The various symbols in the above equations are defined as follows:†

ρ^* = the mass density of the fluid;
ϕ = the magnitude (or length) of the director $\mathbf{d}$;
g = the constant gravitational acceleration;
β = the vertical location of the top fluid surface relative to a fixed system of Cartesian co-ordinate axes (x, y, z);
$\hat{p}$ = pressure in the fluid at its top surface;
$\bar{p}$ = pressure in the fluid at its bottom surface;
p = the Lagrange multiplier, i.e. an arbitrary function of position and time;

also, the fixed level bottom is specified by

$$\bar{\mathbf{p}} = x\mathbf{e}_1, \tag{2.8}$$

and the top surface by

$$\hat{\mathbf{p}} = x\mathbf{e}_1 + \beta(x,t)\,\mathbf{e}_3, \tag{2.9}$$

where, in obtaining (2.8), without loss in generality, we have identified the fixed level bottom of the fluid with $z = 0$ in the x, z plane.

For steady-state motions, the equations of motion (2.6) reduce to

$$(\phi u)_x = 0, \tag{2.10}$$

$$\rho^*\phi u u_x = \hat{p}\beta_x - p_x, \quad \rho^*\phi u\lambda_x = -\rho^* g\phi + \bar{p} - \hat{p}, \quad \tfrac{1}{12}\rho^*\phi u w_x = -\tfrac{1}{2}(\hat{p}+\bar{p}) + p/\phi. \tag{2.11a, b, c}$$

† Our notation for the gravitational acceleration is different from that of Green & Naghdi (1976*a*, 1977) who employed g^* instead of g used here. Also the mass density of the fluid in the three-dimensional theory is designated as ρ^* in order to maintain continuity with previous work (Green & Naghdi 1976*a*, 1976*b*, 1977) in which the symbol ρ is used to denote the mass density per unit area of the fluid sheet. The latter is related to ρ^* by $(50)_1$ of Green & Naghdi (1977).

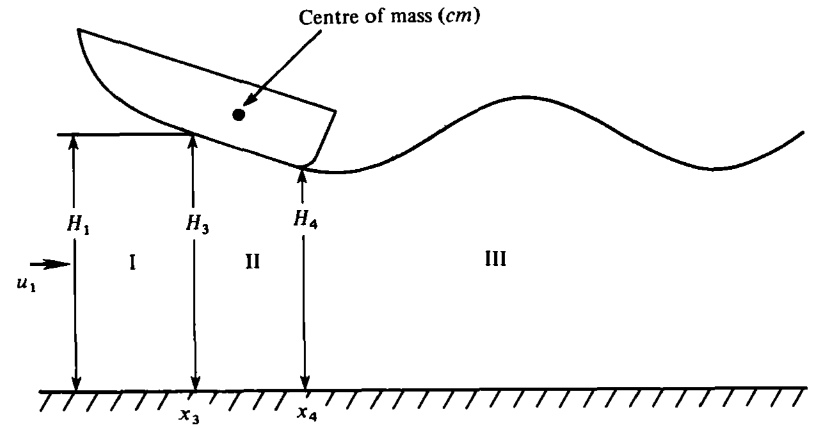

FIGURE 1. A sketch of fluid flow past a boat-like body showing different regions referred to in the text.

Green & Naghdi (1976*b*) have shown that the theory which results in the system of equations (2.6) can be derived from the three-dimensional theory by employing the kinematic assumption that

$$\mathbf{p} = \mathbf{r} + \theta^3 \phi \mathbf{e}_3, \tag{2.12}$$

where θ^3 is a convected co-ordinate. Corresponding to the values $\theta^3 = \pm \frac{1}{2}$ the expression (2.12) locates the top and bottom surfaces, respectively. With the help of (2.8), (2.9) and (2.12), it can then be shown that for a level bottom the quantities ψ and ϕ are related to β by

$$\psi = \tfrac{1}{2}\beta = \tfrac{1}{2}\phi, \quad \phi = \beta. \tag{2.13a, b}$$

As already noted by Green & Naghdi (1977), in the context of the restricted theory of a directed surface, the system of equations (2.6) can be solved independently of the system (2.7). The latter may be satisfied by a specification of the arbitrary quantities s and S.

3. The motion of a fluid past a boat-like body

Consider a boat moving in the x direction with a constant horizontal speed over a body of water whose bottom is level. In describing and analysing the motion of the boat, it is mathematically more convenient to consider the equivalent problem of steady motion of a fluid past a boat-shaped body (see figure 1). Thus, far ahead of the boat the fluid is assumed to flow as a uniform stream; and, in general, at the boat's leading edge (defined by $x = x_3$ in figure 1) there may be some back flow of fluid forming a spray. In addition, the fluid separates smoothly from the boat's trailing edge (defined by $x = x_4$ in figure 1), forming a standing wave. Given the boat's characteristics such as its shape, its weight and suitable information concerning a model of the propulsion system, e.g. the line of action of an *equipolent* propulsion force, our objective is to determine (a) its orientation relative to the fixed bottom of water, (b) the propulsion force required to maintain its speed and (c) the shape of the free surfaces both ahead of and behind the boat.

Although the boat drag cannot be completely described by treating the fluid as inviscid, we expect that the predicted orientation of the boat and the wave pattern behind it will be accurately described by the inviscid theory. At present, the complexities of the boat problem are so great that the exact analysis of the problem via the nonlinear three-dimensional equations is prohibitive. In view of these difficulties and since it is still important to obtain some information about the mechanisms (or the primitive physical bases) which characterize the motion of a boat, it is natural to consider alternatives to the three-dimensional theory. As already noted in §1, here we utilize the equations of the theory of directed fluid sheets summarized in §2 and model the main features of the physical problem under consideration as a fluid sheet flowing past a boat-shaped body (see figure 1). Furthermore, we confine attention to boat shapes and motions for which the hull can be specified uniquely by the function $\beta = \phi$ representing the vertical location of the top surface. This restriction implies that the hull is never vertical. For definiteness, we also require the curvature of the hull $\beta_{xx} = \phi_{xx}$ to be nonnegative. [If the hull curvature ϕ_{xx} is negative, then cavitation pockets (i.e. small regions bounded by points at which the fluid loses and regains contact with the boat's hull) may occur under the hull. This would require subdividing the fluid medium into more regions than are considered here.]

In the analysis of the above problem, it becomes necessary to consider two types of fluid regions: (1) one for which the top surface is free and on which $\hat{p}$ equals the constant ambient pressure p_0; and (2) another for which the top surface is specified and on which $\hat{p}$ is unknown. First, we specialize the system of differential equations (2.10)–(2.11) to one applicable to a fluid region in which the top surface is free. Thus, after setting $\hat{p} = p_0$, we may integrate (2.10) and then rewrite the system of equations (2.10)–(2.11) as

$$\phi u = k, \qquad \rho^* k^2 \left(\frac{1}{\phi}\right)_x = -P_x, \tag{3.1a, b}$$

$$\tfrac{1}{2}\rho^* k^2 \left(\frac{\phi_x}{\phi}\right)_x = -\rho^* g\phi + \bar{P}, \quad \tfrac{1}{12}\rho^* k^2 \left(\frac{\phi_x}{\phi}\right)_x = -\tfrac{1}{2}\bar{P} + \frac{P}{\phi}, \tag{3.1c, d}$$

where in obtaining (3.1) use has been made of (2.12), k is a constant of integration representing the volumetric flow rate per unit width and we have introduced the notations

$$P = p - p_0\phi, \quad \bar{P} = \bar{p} - p_0. \tag{3.2a, b}$$

Integration of (3.1*b*) at once yields

$$P = \rho^*[S - k^2/\phi], \tag{3.3}$$

where S is a constant of integration. [The constant of integration in (3.3) and elsewhere in the paper need not be confused with the temporary use, for a different purpose, of the same symbol in §2 and in the appendix.] Also, from (3.1*c*) we have

$$\bar{P} = \rho^* \left[g\phi + \tfrac{1}{2}k^2 \left(\frac{\phi_x}{\phi}\right)_x\right]. \tag{3.4}$$

Next, we substitute (3.3) and (3.4) into (3.1*d*), multiply the result by ϕ_x/ϕ and integrate to obtain

$$\tfrac{1}{3}k^2\phi_x^2 = -g\phi^3 + 2R\phi^2 - 2S\phi + k^2, \tag{3.5}$$

where R is another constant of integration. The last differential equation may be integrated in terms of an elliptic integral of the first kind (see von Kármán & Biot 1940, p. 121) and the general problem of steady two-dimensional motion over a level bottom is seen to be soluble when the upper surface is free.

We now consider the second type of region mentioned above and specialize the system of differential equations (2.10)–(2.11) to one applicable to a fluid region in which the top surface is specified. In this case, $\hat{p}$ is an unknown and must be determined as part of the solution. Again, we use (2.12), integrate (2.10) and rewrite the system of equations (2.10)–(2.11) as

$$\phi u = k, \quad \rho^* k^2 \left(\frac{1}{\phi}\right)_x = \hat{p}\phi_x - p_x, \tag{3.6a, b}$$

$$\tfrac{1}{2}\rho^* k^2 \left(\frac{\phi_x}{\phi}\right)_x = -\rho^* g\phi + \overline{p} - \hat{p}, \quad \tfrac{1}{12}\rho^* k^2 \left(\frac{\phi_x}{\phi}\right)_x = -\tfrac{1}{2}(\hat{p}+\overline{p}) + \frac{p}{\phi}. \tag{3.6c, d}$$

The expressions for $\overline{p}$ and p from (3.6c) and (3.6d) are seen to be

$$\overline{p} = \hat{p} + \rho^* \left[g\phi + \tfrac{1}{2}k^2 \left(\frac{\phi_x}{\phi}\right)_x\right], \quad p = \hat{p}\phi + \rho^* \left[\tfrac{1}{2}g\phi^2 + \tfrac{1}{3}k^2\phi \left(\frac{\phi_x}{\phi}\right)_x\right]. \tag{3.7a, b}$$

Substituting (3.7b) into (3.6b), dividing by ϕ and integrating, we deduce that

$$\hat{p} = p_0 + \rho^* \left[B - g\phi - \frac{1}{2}\frac{k^2}{\phi^2}(1 - \tfrac{1}{3}\phi_x^2) - \tfrac{1}{3}k^2 \frac{\phi_{xx}}{\phi}\right], \tag{3.8}$$

where B is a constant of integration. Also, substitution of (3.8) into (3.7a, b) yields

$$\overline{p} = p_0 + \rho^* \left[B - \frac{1}{2}\frac{k^2}{\phi^2} - \tfrac{1}{3}k^2 \left(\frac{\phi_x}{\phi}\right)^2 + \tfrac{1}{6}k^2 \frac{\phi_{xx}}{\phi}\right], \tag{3.9a}$$

$$p = p_0\phi + \rho^* \left[B\phi - \tfrac{1}{2}g\phi^2 - \frac{k^2}{2\phi}(1 + \tfrac{1}{3}\phi_x^2)\right]. \tag{3.9b}$$

Since the function ϕ is specified in the region under discussion, it is clear that all kinematical and kinetical quantities are determined once B and k are known.

In addition to the two systems of equations (3.1) and (3.6), we can derive a Bernoulli-type integral for the steady motion of a fluid of variable initial depth when the top surface is either specified or is free. Multiplying (2.11a) by u, (2.11b) by λ, (2.11c) by w, adding the results and using (2.10) and (2.13), we may integrate to obtain

$$\rho^* \phi u[\tfrac{1}{2}(u^2 + \tfrac{1}{3}w^2 + 2\phi)] + pu = C, \tag{3.10}$$

where C is a constant of integration. [It may be noted that, since in the above development $\hat{p}$ is an arbitrary function of x and t, the results (3.8), (3.9) and (3.10) hold even if surface tension is included. A Bernoulli integral of the type (3.10) when the surface tension is included and the top surface is free (rather than specified) was given previously (Green & Naghdi 1976b).]

In the context of the present direct approach, the boat problem can be analysed by considering separately the three regions labelled I, II, III in figure 1. In regions I and III, the top surface is taken to be free and in region II the top surface is specified in the sense that it is in contact with a free-floating (or a fixed) body. The system of equations (3.1a), (3.3), (3.4) and (3.5) describe the flow in regions I and III, while

(3.6a), (3.8) and (3.9) represent the solution for the flow in region II. As indicated above, the solution of (3.5) for ϕ is expressible in terms of an elliptic integral of the first kind; and, then, the remaining quantities u, P and $\bar{P}$ can be calculated from (3.1a), (3.3) and (3.4). Hence, in the context of the theory utilized here, in order to obtain a uniformly valid solution for the whole region $-\infty < x < \infty$, we need to consider appropriate conditions to match the solutions in regions I, II, III. This matching will be accomplished by using the jump conditions associated with the integral balance laws of the basic theory. But before introducing these matching conditions, special attention must be given to the boat's leading and trailing edges.

Due to the two-dimensional character of the theory utilized, we anticipate that at the leading edge of the boat the depth ϕ will be continuous but the slope ϕ_x of the top surface may be discontinuous (see figure 1). In the event that ϕ_x is discontinuous at the leading edge, an examination of (3.8) and (3.9a) reveals that the pressures $\hat{p}$ and $\bar{p}$ may be unbounded there, while p will remain bounded. Keeping this background in mind, in conjunction with the equations of motion (3.6) we introduce the resultants

$$F_1 = \lim_{\delta\to 0}\int_{x_3-\delta}^{x_3+\delta} \hat{p}\phi_x dx, \quad F_3 = \lim_{\delta\to 0}\int_{x_3-\delta}^{x_3+\delta} (\bar{p}-\hat{p})dx, \tag{3.11a, b}$$

$$L_3 = \lim_{\delta\to 0}\int_{x_3-\delta}^{x_3+\delta} [-\tfrac{1}{2}(\hat{p}+\bar{p})]dx, \tag{3.11c}$$

which are similar to those employed by Caulk (1976) in a different context. It will be shown later that the resultants (3.11) can be used in certain situations to model the force that the 'spray root' exerts on the boat's leading edge.

We now recall from the appendix of this paper the appropriate form of the jump conditions (A 9) needed to match the solutions on either side of the leading edge. Thus, with the mass density ρ^* continuous, we have

$$[\![\phi u]\!]_3 = 0, \tag{3.12a}$$

$$[\![\rho^*\phi u^2+p]\!]_3 = F_1, \quad [\![\tfrac{1}{2}\rho^*\phi uw]\!]_3 = F_3, \quad [\![\tfrac{1}{12}\rho^*\phi uw]\!]_3 = L_3, \tag{3.12b, c, d}$$

$$[\![\tfrac{1}{2}\rho^*\phi u(u^2+\tfrac{1}{3}w^2+g\phi)+pu]\!]_3 = -\Phi. \tag{3.12e}$$

In the above formulae, the notation $[\![f]\!]_3$ stands for the jump in f across $x = x_3$, i.e.

$$[\![f]\!]_3 = f_3^+ - f_3^- = f|_{x=x_3^+} - f|_{x=x_3^-}, \tag{3.13}$$

and the function Φ on the right-hand side of (3.12e), which requires a constitutive equation, represents the rate of dissipation of energy per unit length introduced in the energy jump given by the last of (A 4). We also note that the jump condition associated with angular momentum is identically satisfied in view of the fact that $\mathbf{r}$ and $\mathbf{d}$ in (2.3) are assumed to be continuous and hence

$$[\![\phi]\!]_3 = 0. \tag{3.14}$$

Since the fluid is assumed to separate smoothly from the boat's trailing edge, the slope ϕ_x of the top surface of the fluid sheet remains continuous. Consequently, the pressures $\hat{p}$ and $\bar{p}$ are bounded there and the resultants corresponding to (3.11a, b, c) vanish. Therefore, in the absence of the resultants, from (3.12) the matching conditions associated with the trailing edge are

$$[\![\phi u]\!]_4 = 0, \tag{3.15a}$$

$$[\![\rho^*\phi u^2+p]\!]_4 = 0, \quad [\![\tfrac{1}{2}\rho^*\phi uw]\!]_4 = 0, \quad [\![\tfrac{1}{12}\rho^*\phi uw]\!]_4 = 0, \tag{3.15b, c, d}$$

$$[\![\tfrac{1}{2}\rho^*\phi u(u^2+\tfrac{1}{3}w^2+g\phi)+pu]\!]_4 = 0, \tag{3.15e}$$

where again the condition associated with angular momentum is identically satisfied since

$$[\![\phi]\!]_4 = 0 \tag{3.16}$$

and where the notation (3.13) has been used with x_3 replaced by x_4. It should be noted that not all conditions in (3.15) and (3.16) are independent. In particular, with the help of (3.15*a*, *b*, *c*) and (3.16), the jump in director momentum (3.15*d*) and energy (3.15*e*) are identically satisfied.

It is convenient to summarize here the equations and the solutions associated with the three regions in the boat problem.

Region I:

$$\phi u = k_1, \tag{3.17a}$$

$$P = p-p_0\phi = \rho^*\left[S_1-\frac{k_1^2}{\phi}\right], \quad \bar{P} = \bar{p}-p_0 = \rho^*\left[g\phi+\tfrac{1}{2}k_1^2\left(\frac{\phi_x}{\phi}\right)_x\right], \tag{3.17b, c}$$

$$\tfrac{1}{3}k_1^2\phi_x^2 = -g\phi^3+2R_1\phi^2-2S_1\phi+k_1^2, \tag{3.17d}$$

$$C_1 = \tfrac{1}{2}\rho^*\phi u(u^2+\tfrac{1}{3}w^2+g\phi)+pu = \rho^*k_1R_1+p_0k_1. \tag{3.17e}$$

Region II:

$$\phi u = k_2, \tag{3.18a}$$

$$p = p_0\phi+\rho^*\left[B\phi-\tfrac{1}{2}g\phi^2-\frac{1}{2}\frac{k_2^2}{\phi}(1+\tfrac{1}{3}\phi_x^2)\right], \tag{3.18b}$$

$$\hat{p} = p_0+\rho^*\left[B-g\phi-\frac{1}{2}\frac{k_2^2}{\phi^2}(1-\tfrac{1}{3}\phi_x^2)-\tfrac{1}{3}k_2^2\frac{\phi_{xx}}{\phi}\right], \tag{3.18c}$$

$$\bar{p} = p_0+\rho^*\left[B-\frac{1}{2}\frac{k_2^2}{\phi^2}-\tfrac{1}{3}k_2^2\left(\frac{\phi_x}{\phi}\right)^2+\tfrac{1}{6}k_2^2\frac{\phi_{xx}}{\phi}\right], \tag{3.18d}$$

$$C_2 = \tfrac{1}{2}\rho^*\phi u(u^2+\tfrac{1}{3}w^2+g\phi)+pu = \rho^*k_2B+p_0k_2. \tag{3.18e}$$

Region III:

$$\phi u = k_3, \tag{3.19a}$$

$$P = p-p_0\phi = \rho^*\left[S_3-\frac{k_3^2}{\phi}\right], \quad \bar{P} = \bar{p}-p_0 = \rho^*\left[g\phi+\tfrac{1}{2}k_3^2\left(\frac{\phi_x}{\phi}\right)_x\right], \tag{3.19b, c}$$

$$\tfrac{1}{3}k_3^2\phi_x^2 = -g\phi^3+2R_3\phi^2-2S_3\phi+k_3^2, \tag{3.19d}$$

$$C_3 = \tfrac{1}{2}\rho^*\phi u(u^2+\tfrac{1}{3}w^2+g\phi)+pu = \rho^*k_3R_3+p_0k_3. \tag{3.19e}$$

In (3.17)–(3.19), k_1, k_2, k_3, C_1, C_2, C_3, S_1, S_3, R_1, R_3 and B are all constants. Once the points x_3 and x_4 are determined, the conditions (3.12) and (3.14)–(3.16) can be used to match the solutions in regions I, II and III to obtain a uniformly valid solution for the whole region.

With the help of (3.17), (3.18), (3.19) and the matching conditions (3.12), (3.14), (3.15) and (3.16), we combine the solutions in regions I–III to obtain

$$k_1 = k_2 = k_3, \quad \phi_3^- = \phi_3^+, \quad \phi_4^- = \phi_4^+, \quad \phi_{4x}^- = \phi_{4x}^+, \tag{3.20a, b, c, d}$$

$$\Phi = \rho^* k(R_1 - B), \quad R_3 = B, \tag{3.20e, f}$$

$$S_3 = BH_4 - \tfrac{1}{2}gH_4^2 + \frac{1}{2}\frac{k^2}{H_4} - \tfrac{1}{6}k^2\frac{K_4^2}{H_4}, \tag{3.20g}$$

$$F_1 = \tfrac{1}{6}\rho^* \frac{k^2}{H_3}(K_2^2 - K_3^2) - \frac{\Phi H_3}{k}, \quad F_3 = \tfrac{1}{2}\rho^* \frac{k^2}{H_3}(K_3 - K_2), \tag{3.20h, i}$$

$$L_3 = \tfrac{1}{12}\rho^* \frac{k^2}{H_3}(K_3 - K_2), \tag{3.20j}$$

where for convenience we have introduced the additional notations

$$k_1 = k, \quad \phi_3^- = H_3, \tag{3.21a, b}$$

$$\phi_{3x}^- = K_2, \quad \phi_{3x}^+ = K_3, \quad \phi_4^- = H_4, \quad \phi_{4x}^- = K_4, \tag{3.21c, d, e, f}$$

and have evaluated S_3 in (3.20g) by using (3.19d), (3.20) and (3.21). When k vanishes, it follows from (3.8), (3.11a), (3.20e) and (3.20h) that $F_1 = 0$ and $B = R_1$.

Next, we turn our attention to the boundary conditions. Recalling that far ahead of the boat we have assumed the fluid to flow as a uniform stream, we impose the conditions that

$$\phi \to H_1, \quad \phi_x \to 0, \quad p \to p_0 H_1 + \tfrac{1}{2}\rho^* g H_1^2, \quad u \to u_1 \quad \text{as} \quad x \to -\infty, \tag{3.22a, b, c, d}$$

where H_1 and u_1 are constants and we require $u_1 \geqslant 0$, since we are concerned here with fluid flow towards the bow of the boat. With the help of (3.17) and (3.22) we conclude that

$$k_1 = H_1 u_1, \quad S_1 = \tfrac{1}{2}gH_1^2 + \frac{k_1^2}{H_1}, \quad R_1 = gH_1 + \frac{1}{2}\frac{k_1^2}{H_1^2} \tag{3.23a, b, c}$$

and that

$$\tfrac{1}{3}k_1^2\phi_x^2 = (H_1 - \phi)^2\left(\frac{k_1^2}{H_1^2} - g\phi\right). \tag{3.24}$$

Introducing the dimensionless quantities

$$\bar{\phi} = \frac{\phi}{H_1}, \quad \bar{x} = \frac{x}{H_1} \tag{3.25}$$

and the Froude number F associated with the equilibrium depth H_1,

$$F^2 = \frac{u_1^2}{gH_1}, \tag{3.26}$$

the differential equation (3.24) and the condition (3.22a) can be written in the form

$$\tfrac{1}{3}F^2\bar{\phi}_{\bar{x}}^2 = (1 - \bar{\phi})^2(F^2 - \bar{\phi}), \quad \bar{\phi} \to 1 \quad \text{as} \quad \bar{x} \to -\infty. \tag{3.27a, b}$$

If we confine attention to speeds u_1 for which $F < 1$, then from (3.27) the only solution of (3.24) consistent with the boundary condition (3.22a) is the uniform stream solution

$$\phi = H_1. \tag{3.28}$$

Before closing this section we discuss the conditions used to determine the location of the boat's leading and trailing edges. Since the depth is continuous, the location x_3 of the boat's leading edge is determined by solving the equation

$$\phi|_{x=x_3^+} = H_3. \tag{3.29}$$

If the speed is such that $F < 1$, then from (3.28) we have $H_3 = H_1$ which is specified. On the other hand, if $F \geqslant 1$, then (3.28) is a possible solution of (3.27) but it is not necessarily the only solution. In such cases, additional information is required in order to determine the value of H_3, the location of the boat's leading edge and the appropriate solution for the free surface in the leading region. Throughout this analysis we have assumed that the fluid leaves the boat's trailing edge smoothly, so that at the trailing edge the matching conditions are given by (3.15) and (3.16). Although these conditions require continuity of various quantities, they do not provide any statement about $\hat{p}$, the pressure in the fluid at its top surface. Physically we expect the fluid to maintain contact with the boat's trailing edge until the pressure there becomes equal to the atmospheric pressure, at which point the fluid separates from the boat. The boat's trailing edge, located by $x = x_4$, is then determined by requiring the pressure $\hat{p}$ to be equal to p_0.† Thus, in line with these observations and with the help of (3.18*c*), (3.20*a*) and (3.21*a*), we impose the following condition

$$\left[B - g\phi - \frac{1}{2}\frac{k^2}{\phi^2}(1 - \tfrac{1}{3}\phi_x^2) - \tfrac{1}{3}k^2\frac{\phi_{xx}}{\phi}\right]\Bigg|_{x=x_4} = 0, \tag{3.30}$$

from which x_4 can be determined. Once the quantities H_1, u_1, H_3 and Φ are specified and the differential equations (3.17*d*) and (3.19*d*) are integrated subject to the conditions (3.21*b*) and (3.21*e*, *f*), respectively, the complete flow past a given boat-like body is determined.

4. Equations governing the motion of a free-floating boat-like body

We first consider here the effect of the fluid motion on a boat-like body whose hull geometry has continuous curvature, and then formulate the coupled problem in which the boat adjusts its orientation according to the fluid motion. Let $\hat{\mathbf{r}}$ and $\mathbf{x}_{cm}$ denote, respectively, the position vector to points on the boat's bottom surface and the position vector of the centre of mass of the boat. Then, referred to base vectors $\mathbf{e}_1$ and $\mathbf{e}_3$, these quantities can be written as

$$\hat{\mathbf{r}} = x\mathbf{e}_1 + \phi\mathbf{e}_3 \tag{4.1}$$

and

$$\mathbf{x}_{cm} = x_{cm}\mathbf{e}_1 + z_{cm}\mathbf{e}_3. \tag{4.2}$$

The net force $\mathscr{F}$ and the net moment $\mathscr{M}$ (about the centre of mass) acting on the boat's bottom, each per unit length in the y direction, arise from the pressure difference $\hat{p} - p_0$ and are defined by

$$\mathscr{F} = -\int_{x_3^-}^{x_4} (\hat{p} - p_0)(\phi_x\mathbf{e}_1 - \mathbf{e}_3)\,dx, \tag{4.3a}$$

$$\mathscr{M} = -\int_{x_3^-}^{x_4} (\hat{\mathbf{r}} - \mathbf{x}_{cm}) \times (\hat{p} - p_0)(\phi_x\mathbf{e}_1 - \mathbf{e}_3)\,dx. \tag{4.3b}$$

† A condition similar to this was employed by Keller & Weitz (1957) in a different context. After this paper was completed, it was brought to our attention that a similar condition was also used by Haussling & Van Eseltine (1976) in their numerical solution of planing-body problems.

[In the literature on naval architecture pertaining to ship design, these quantities are often referred to as forces 'per unit beam'.] Remembering that the pressure $\hat{p}$ may become unbounded at the boat's leading edge, it is convenient to define the total drag $\mathscr{D}_T$, the total lift $\mathscr{L}_T$ and the total moment $\mathscr{M}_T$ (about the boat's centre of mass) and separate each of these into two parts: one part identified with a subscript R (e.g. $\mathscr{D}_R$) arising from the unboundedness of $\hat{p}$ at the boat's leading edge and the second part identified with a subscript P (e.g. $\mathscr{D}_P$) due to the integrated effect of the pressure difference $\hat{p}-p_0$ exerted by the fluid on the boat's hull. Thus, we introduce the following definitions:

$$\mathscr{D}_T = \mathscr{D}_R+\mathscr{D}_P, \quad \mathscr{L}_T = \mathscr{L}_R+\mathscr{L}_P, \quad \mathscr{M}_T = \mathscr{M}_R+\mathscr{M}_P, \tag{4.4a, b, c}$$

$$\mathscr{D}_R = -\lim_{\delta\to 0}\int_{x_3-\delta}^{x_3+\delta}(\hat{p}-p_0)\phi_x\,dx, \quad \mathscr{D}_P = -\int_{x_3^+}^{x_4}(\hat{p}-p_0)\phi_x\,dx, \tag{4.4d, e}$$

$$\mathscr{L}_R = \lim_{\delta\to 0}\int_{x_3-\delta}^{x_3+\delta}(\hat{p}-p_0)dx, \quad \mathscr{L}_P = \int_{x_3}^{x_4}(\hat{p}-p_0)dx, \tag{4.4f, g}$$

$$\mathscr{M}_R = -\lim_{\delta\to 0}\int_{x_3-\delta}^{x_3+\delta}(\hat{p}-p_0)\,[(x-x_{cm})+(\phi-z_{cm})\phi_x]dx, \tag{4.4h, i}$$

$$\mathscr{M}_P = -\int_{x_3^+}^{x_4}(\hat{p}-p_0)\,[(x-x_{cm})+(\phi-z_{cm})\phi_x]\,dx, \tag{4.4j}$$

and then write

$$\mathscr{F} = \mathscr{D}_T\mathbf{e}_1+\mathscr{L}_T\mathbf{e}_3, \quad \mathscr{M} = \mathscr{M}_T\mathbf{e}_2. \tag{4.5a, b}$$

Since p_0 and ϕ_x are bounded and ϕ is continuous, with the use of (3.11) and (4.4) the quantities $\mathscr{D}_R$, $\mathscr{L}_R$ and $\mathscr{M}_R$ may be represented as

$$\mathscr{D}_R = -F_1, \quad \mathscr{L}_R = -\tfrac{1}{2}(F_3+2L_3), \tag{4.6a, b}$$

$$\mathscr{M}_R = \tfrac{1}{2}(x_3-x_{cm})(F_3+2L_3)-(H_3-z_{cm})F_1, \tag{4.6c}$$

where in obtaining (4.6) use is made of the fact that

$$\tfrac{1}{2}(F_3+2L_3) = -\lim_{\delta\to 0}\int_{x_3-\delta}^{x_3+\delta}\hat{p}\,dx. \tag{4.7}$$

Since the terms $\mathscr{D}_R$, $\mathscr{L}_R$, $\mathscr{M}_R$ in (4.6), which involve resultants F_1, F_3, L_3, arise from the unboundedness of the pressure $\hat{p}$, we observe that in certain situations they may be used to model the effects of the spray root on the boat's leading edge. In such situations we identify the resultant drag $\mathscr{D}_R$ with the horizontal drag due to the spray root and write†

$$\mathscr{D}_R = -F_1 = \tfrac{1}{6}\rho^*\frac{k^2K_3^2}{H_1}+\frac{\Phi H_1}{k}, \tag{4.8}$$

where use has been made of (3.20h). It should be noted that in writing (4.8) we have restricted attention to situations for which the solution (3.28) holds. Recalling that the only energy-dissipating mechanism in this flow is that of spray formation, the

† It will be seen in §§5 and 7 that, in the examples of the transition to planing of a boat or a wedge, the identification of $\mathscr{D}_R$, $\mathscr{L}_R$, $\mathscr{M}_R$ with the effects of the spray root is reasonable. However, in other situations such an identification might not be desirable.

function Φ (which in general requires a constitutive equation) on the right-hand side of (4.8) can be used to model this energy loss. Physically we require the energy lost to spray formation to be nonnegative (i.e. $\Phi \geqslant 0$), so that by (4.8) the spray drag increases with increased dissipation (or energy loss). In this connexion, it may be recalled that, in the present formulation of the problem, the fluid flows towards the bow of the boat so that $u_1 \geqslant 0$. Moreover, by (3.20*e*) and the remark made following (3.21), it is clear that $\Phi = 0$ and $\mathscr{D}_R = 0$ when k vanishes.

Although we have already calculated the drag $\mathscr{D}_T$, it is possible to avoid evaluating the integrals in (4.4*d*, *e*) by using an alternative procedure. Thus, with the help of (3.6*b*), (3.17*b*), (3.19*b*) and (4.4*a*, *d*, *e*) we have

$$\mathscr{D}_T = -\left[\rho^* \frac{k^2}{\phi} + p - p_0 \phi\right]_{x_3^-}^{x_4} = \rho^*(S_1 - S_3), \tag{4.9}$$

where S_1 and S_3 are determined by (3.23*b*) and (3.20*g*), respectively.

Before considering the equations of motion of the boat, it is necessary to characterize its shape and propulsion system. To this end, consider a set of body co-ordinates $x_i' = (x', y', z')$ with base vectors $\mathbf{e}_i' = (\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3')$ fixed in the boat such that the origin of the co-ordinate system coincides with the boat's centre of mass. Let $\mathbf{F}_p$ be the propulsion force acting at the point $\mathbf{r}_p$ located relative to the boat's centre of mass and let W be the weight of the boat, both measured per unit length along the y direction. Since $\mathscr{F}$ and $\mathscr{M}$ are the net force and moment of the fluid on the boat and since the boat is regarded as a rigid body in equilibrium, the governing equations are

$$\mathscr{F} - W\mathbf{e}_3 + \mathbf{F}_p = 0, \quad \mathscr{M} + \mathbf{r}_p \times \mathbf{F}_p = 0, \tag{4.10a, b}$$

where

$$\mathbf{e}_1' = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_3, \quad \mathbf{e}_3' = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_3, \tag{4.11a, b}$$

$$\mathbf{r}_p = x_p'\mathbf{e}_1' + z_p'\mathbf{e}_3', \quad \mathbf{F}_p = -F_p(\cos\gamma\mathbf{e}_1' + \sin\gamma\mathbf{e}_3') \tag{4.11c, d}$$

and where θ and γ are measured positive in the counterclockwise direction (see figure 2). Further, let the location of the boat's bottom relative to its centre of mass be specified by

$$\mathbf{r}' = x'\mathbf{e}_1' + \eta(x')\mathbf{e}_3', \tag{4.12}$$

where η is a function of x' only. For definiteness, we confine attention to boats whose hulls are convex and in order to ensure that the pressure $\hat{p}$ is continuous we require η in (4.12) to be twice continuously differentiable. Also, since the boat is stationary, without loss in generality we may set

$$x_{cm} = 0. \tag{4.13}$$

Once the characteristics of the boat are specified in terms of

$$\mathbf{r}_p,\ W,\ \gamma,\ \eta, \tag{4.14}$$

and the parameters

$$\rho^*,\ g,\ H_1,\ u_1,\ p_0, \tag{4.15}$$

are given, in addition to the dissipation function Φ, then recalling also that here $H_3 = H_1$, (3.29), (3.30) and (4.10) can be used to solve for the unknowns

$$z_{cm},\ \theta,\ x_3,\ x_4,\ F_p. \tag{4.16}$$

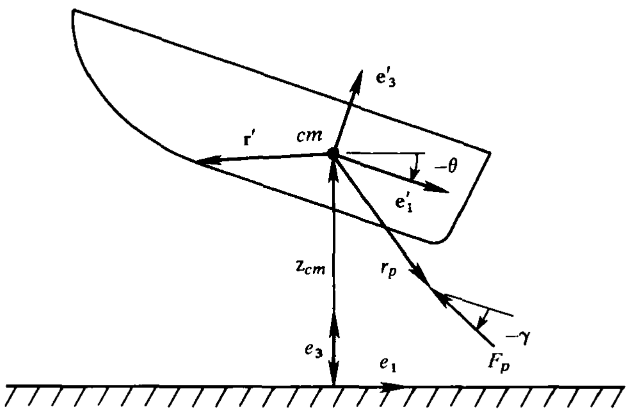

FIGURE 2. Specification of boat's characteristics: (1) the geometry of a boat's hull defined by the position vector $\mathbf{r}'$; (2) the boat's orientation relative to the level bottom defined by the height z_{cm} of the centre of mass (*cm*) and the angle θ measured counterclockwise from the direction of the base vector $\mathbf{e}_1$ along the level bottom; and (3) the equipolent propulsion system of the boat defined by the propulsion force $\mathbf{F}_p$, the position vector $\mathbf{r}_p$ which locates the point of action of $\mathbf{F}_p$ relative to *cm* and the angle γ measured counterclockwise from the direction of the base vector $\mathbf{e}_1'$ fixed in the boat.

Also, the shape of the free surface in the downstream region can be determined by integrating (3.19*d*) subject to the boundary condition (3.21*f*).

Our discussion here has been concerned with boats whose hull geometries have continuous curvature. This restriction has been motivated mainly by the fact that we have required the pressures $\hat{p}$ and $\bar{p}$ in (3.18*c*, *d*) to be continuous on the hull except possibly at the leading edge of the boat. It is possible to consider hull geometries which are sectionally continuous (one example being a wedge shape), effecting considerable analytical simplifications, but we postpone the discussion of the planing of a wedge until §§6 and 7.

5. Transition to planing of a boat-like body

As an application of the theory of the previous sections, we study here the transition to planing of a free-floating, self-propelled, boat-like body. The solution to this problem is obtained with the aid of a computer program developed to solve the five scalar equations resulting from (3.29), (3.30) and (4.10). Once the function ϕ (which describes the vertical location of points along the boat's bottom) is known or specified, then these equations may be regarded as algebraic or essentially algebraic in that they also involve definite integrals. In our computer program, these integrals are evaluated using a twelve-point (including endpoints) Gaussian quadrature scheme by the subroutine GB and the essentially algebraic equations are solved using a Newton–Raphson iteration procedure by the subroutine SIMEQ. [The subroutine GB is available in the Computer Center of the University of California, Berkeley (U.C.B.) and the subroutine SIMEQ was made available by Professor C. W. Radcliffe of the Department of Mechanical Engineering at U.C.B.]

Although our development is applicable to a fairly general class of hull geometries, as remarked in §3 we confine attention to hull geometries which are convex and have continuous curvature. The latter requirement is motivated by the observation that, within the scope of the theory employed, the pressure $\hat{p}$ acting on the boat's bottom (or its hull) determined from (3.18*c*) is seen to be continuous if the hull shape has continuous curvature (i.e. ϕ_{xx} continuous). Keeping this in mind, we specify the function η in (4.12) to be of the form

$$\eta(x') = \begin{cases} \eta_1 - 1 + e^{B(x'-b)} - B(x'-b) - \dfrac{B^2(x'-b)^2}{2} & \text{for } x' > b, \\ \eta_1 & \text{for } a \leqslant x' \leqslant b, \\ \eta_1 - 1 + e^{-A(x'-a)} + A(x'-a) - \dfrac{A^2(x'-a)^2}{2} & \text{for } x' < a, \end{cases} \tag{5.1}$$

where η_1, A, B, a, b are constants. For the particular example to be considered, these constants and the equilibrium depth H_1 are specified by

$$\eta_1 = -0{\cdot}5\ \mathrm{m}, \quad A = 0{\cdot}8\ \mathrm{m}^{-1}, \quad B = 6{\cdot}0\ \mathrm{m}^{-1}, \tag{5.2a, b, c}$$

$$a = -0{\cdot}9\ \mathrm{m}, \quad b = 1{\cdot}2\ \mathrm{m}, \quad H_1 = 10{\cdot}0\ \mathrm{m}. \tag{5.2d, e, f}$$

In addition to (5.1), the weight W per unit length of the boat-like body, the propulsion angle γ in (4.11*d*) and the co-ordinates x_p', z_p' of the point of action of the propulsion force and the rate of energy dissipation Φ are taken to be

$$W = 7000\ \mathrm{N\,m^{-1}}, \quad \gamma = 0{\cdot}0\ \mathrm{rad}, \quad x_p' = 1{\cdot}2\ \mathrm{m}, \quad z_p' = -0{\cdot}82\ \mathrm{m}, \quad \Phi = 0, \tag{5.3a, b, c, d, e}$$

where all quantities in (5.2) and (5.3) are specified in SI units.

It was noted in §3 that (3.5) may be solved in terms of an elliptic integral of the first kind. While (3.19*d*) may be solved similarly, it is necessary to find the roots of the cubic polynomial on the right-hand side of (3.19*d*) before the resulting integral can be reduced to the Legendre standard form of the elliptic integral of the first kind which is tabulated. However, without actually solving for the wave in the trailing regions, the analytical form of the solution of the differential equation (3.19*d*) can be exploited to determine some of its characteristic features. The stationary values of the function ϕ (i.e. values at which ϕ_x vanishes) occur at the roots ϕ_1, ϕ_2 and ϕ_3 of the cubic polynomial on the right-hand side of (3.19*d*). Recalling the remarks made at the end of §4 [between (4.13) and (4.16)], once the depth H_1, the velocity u_1 and the orientation of the boat are known, we can solve for ϕ_1, ϕ_2 and ϕ_3 and then the character of the wave in the trailing region is also determined. Although it is possible to categorize different cases corresponding to different wave-like characters in the trailing region, we only mention three cases of particular interest here:

Case (1). Let ϕ_1, ϕ_2 and ϕ_3 be real numbers such that $\phi_1 > H_1 > \phi_2 > \phi_3$ and let H_4 be such that $\phi_1 > H_4 \geqslant \phi_2$. Under these circumstances, the height of the free surface in the trailing region oscillates about the equilibrium value H_1 and attains its maximum and minimum values ϕ_1 and ϕ_2, respectively.

Case (2). Let ϕ_1, ϕ_2 and ϕ_3 be real numbers such that two of the roots coalesce $\phi_1 > H_1 > \phi_2 = \phi_3$ and let H_4 and K_4 be such that $\phi_1 > H_4 > \phi_2$ and $K_4 < 0$. Under these circumstances, the wavelength of the wave in the trailing region is infinite and the height of the free surface decreases monotonically from the value H_4 at the trailing edge of the boat to the value ϕ_2 at distances far away from the boat.

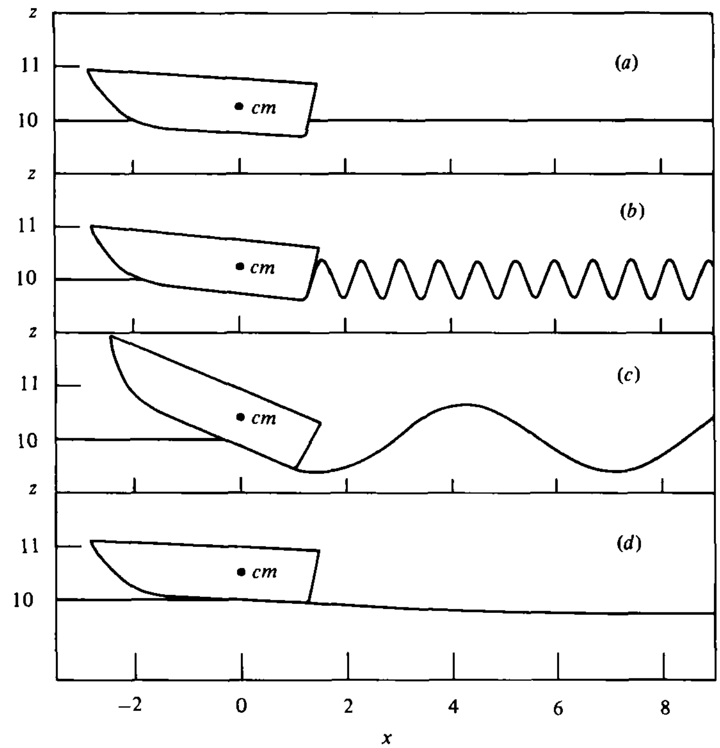

FIGURE 3. Solution of the problem of transition to planing of the boat defined by (5.1)–(5.3) for several configurations corresponding to different depth Froude numbers F: (a) equilibrium configuration, $F = 0$; (b) slow speed, $F = 0{\cdot}0202$; (c) approximate hump speed, $F = 0{\cdot}1514$; and (d) speed of minimum wave amplitude, $F = 0{\cdot}7168$.

Case (3). Let ϕ_1 be a real number $\phi_1 > H_1$ with ϕ_2 and ϕ_3 being complex conjugates of each other and let H_4 and K_4 be such that $\phi_1 > H_4$ and $K_4 < 0$. Under these circumstances, no wave-like solution exists in the trailing region. The height of the free surface monotonically decreases from the value H_4 at the trailing edge of the boat to zero at some finite distance from the boat.

Clearly, in principle, the solution of (3.19d) may be reduced to the Legendre standard form of the elliptic integral of the first kind. But such an analytical reduction is quite cumbersome, and it is more convenient to obtain the solution for the free surface in the trailing region numerically. For this purpose, we solve for $\bar{P}$ and P from (3.1c, d):

$$\bar{P} = \rho^*\left[g\phi + \tfrac{1}{2}k^2\left(\frac{\phi_x}{\phi}\right)_x\right], \quad P = \rho^*\left[\tfrac{1}{2}g\phi^2 + \tfrac{1}{3}k^2\phi\left(\frac{\phi_x}{\phi}\right)_x\right]. \qquad (5.4a, b)$$

Then, from (5.4b) and (3.3) readily follows the following differential equation:

$$\tfrac{1}{3}k^2\phi\phi_{xx} - \tfrac{1}{3}k^2\phi_x^2 + \tfrac{1}{2}g\phi^3 - S\phi + k^2 = 0, \qquad (5.5)$$

which determines the height of the free surface in the trailing region. For regions containing a free top surface, the complete solution is obtained from the solution of (5.5), (5.4) and (3.1a). We observe that the conditions (3.20a, b, c) show that the quantities k, ϕ and ϕ_x are continuous at the boat's trailing edge; and, in the trailing region, the constant S is equated to S_3 which is given by (3.20f). Since ϕ_4^-, its derivative ϕ_{4x}^- and the flow rate k are determined in the course of solving the equations (3.29), (3.30) and (4.10), they can be used to specify the boundary conditions for the solution of the differential equation (5.5), which is solved numerically using a fifth-order Runge–Kutta integration scheme by the subroutine RK5. [The subroutine RK5 is available in the U.C.B. Computer Center.]

The numerical results of the solution for the example considered are presented in figures 3–5. Fig. 3(a–d) displays the solution for the boat and the shape of the free surfaces pictorially for a range of values of the depth Froude number F. As F increases from zero to a critical value, the solution predicts that the wave amplitude first increases to a maximum (close to the value indicated in figure 3(c)), then decreases to a minimum (close to the value in figure 3(d)), and increases again until the critical value of F is attained. For F below the critical value, the solution is described by case (1) discussed above, so that the free surface in the trailing region is wave-like and oscillates about its equilibrium height H_1. At the critical value of F, the two roots ϕ_2 and ϕ_3 coalesce to a single value so that the solution in the trailing region is one in which the free surface descends monotonically to this value (case (2) above). The shape of the free surface just described represents a wave with an infinite wavelength and is reminiscent of the shape of the free surface in the downstream region of a sluice gate. [The interrelationship between the solutions representing a sluice-like and boat-like behaviour in the trailing region was also noted by Benjamin (1956) in a different context.] As the Froude number increases above the critical value, the solution is described by case (3) above, so that the free surface in the trailing region can no longer oscillate about its equilibrium height. Consequently, for speeds corresponding to Froude numbers greater than this critical value, which for the numerical example under discussion is approximately $F = 0{\cdot}9491$, no wave-like solution is possible.†

In his analysis of a wedge-shaped boat, Squire (1957) assumes that the flow separates from the wedge's bottom edge at all speeds above zero. In the context of the direct approach of this paper, such an assumption would require abandoning the condition that the pressure be atmospheric on the boat's bottom at its trailing edge and is further discussed in §7. By contrast, in the present formulation of the problem, the location of the trailing edge is determined by the nonlinear condition (3.30). A consequence of this condition is shown in figure 3(b), where, at slow speeds, the flow separates from the hull at a point on the back of the boat.

The hump speed is usually defined to be the speed at which the boat experiences maximum 'resistance' to motion (see for example Saunders (1957)). Although in this definition the term 'resistance' is somewhat vague and can be taken to mean either the horizontal drag on the boat or the boat's propulsion force, the latter interpretation has been employed for the presentation of the numerical results in figure 3(c). In order to predict the hump speed phenomenon, Squire (1957) introduces a frictional drag

† The existence of a critical speed beyond which no wave-like solutions exist is not entirely unfamiliar even in the three-dimensional theory of an inviscid incompressible fluid in irrotational motions. If this three-dimensional theory is linearized about the two-dimensional uniform stream solution, then it can be shown that no wave-like solutions exist for supercritical Froude numbers ($F > 1$). This same result may also be derived using the theory of fluid sheets of this paper. We note, however, that the solution presented here is nonlinear so that one should not expect the critical value of F to be exactly unity.

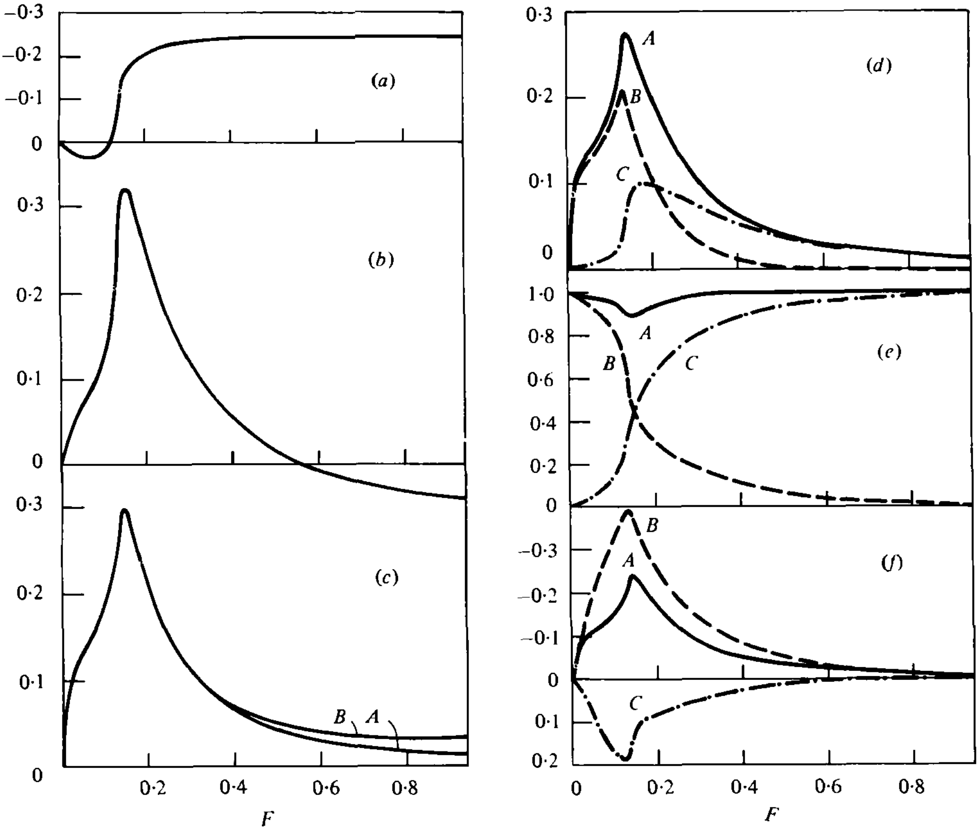

FIGURE 4. (*a*, *b*, *c*) Transition to planing of a boat, several aspects of the solution: (*a*) the sinkage $\Delta z = z_{cm0} - z_{cm}$ in metres, where $z_{cm0} = 10{\cdot}25420$ m; (*b*) the bow-up trim angle $\Delta\theta = \theta_0 - \theta$ in radians, where $\theta_0 = -0{\cdot}064262$ rad; and (*c*) the normalized propulsion force F_p/W, where curves A and B represent the inviscid and viscous results, respectively, and where the weight of the boat $W = 7000\ \mathrm{Nm}^{-1}$. (*d*, *e*, *f*) Transition to planing of a boat, several aspects of the solution pertaining to net forces and moments (about the boat's centre of mass) exerted by the fluid on the boat, normalized by the weight of the boat $W = 7000\ \mathrm{Nm}^{-1}$: (*d*) the normalized total drag $\mathscr{D}_T/W$ (curve A) due to the normalized drag forces $\mathscr{D}_P/W$ (curve B) and $\mathscr{D}_R/W$ (curve C); (*e*) the normalized total lift $\mathscr{L}_T/W$ (curve A) due to the normalized lift forces $\mathscr{L}_P/W$ (curve B) and $\mathscr{L}_R/W$ (curve C); and (*f*) the normalized total bow-up moment $\mathscr{M}_T/W$ (curve A) due to the normalized moments $\mathscr{M}_P/W$ (curve B) and $\mathscr{M}_R/W$ (curve C), each measured in metres.

term and concludes that, for a wedge planing at constant trim angle, the hump speed is determined solely by viscous effects. The numerical results presented in figure 4 for the example of the transition to planing of a boat considered here show that the hump speed can be predicted by a purely inviscid theory. Furthermore, detailed examination of figure 4 suggests that the dominant mechanism governing the hump speed phenomenon is the change in trim angle. In view of this observation, it appears more natural to define the hump speed for planing crafts as the speed at which the bow-up trim angle becomes a maximum. Such a definition is also free of the difficulty that, although the propulsion force reaches a local maximum near the hump speed, it may attain an absolute maximum at a speed far in excess of the hump speed.

Since the wedge considered by Squire (1957) planes at constant trim angle and no

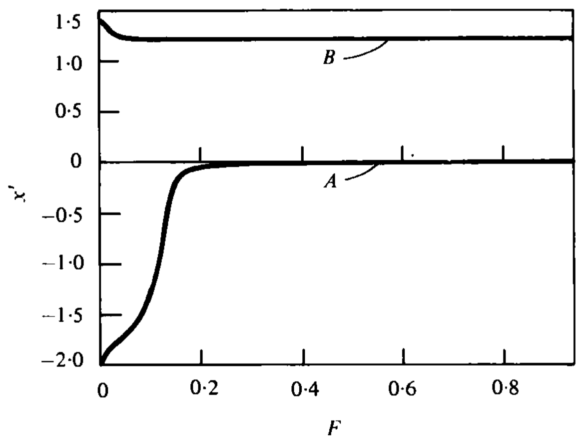

FIGURE 5. Variation of the wetted length defined by (5.6) in the solution for the transition to planing of a boat. The curves A and B represent the values in metres of x'_3 and x'_4 which locate, respectively, the leading and trailing edges of the boat relative to the body co-ordinate x' along the direction of $\mathbf{e}'_1$ shown in figure 2.

resultants are introduced, knowledge of the lift on the wedge and the wedge angle is enough to determine the drag due to the pressure on the wedge's bottom. On the other hand, here we consider general hull shapes, allow the trim angle to change, and introduce resultants which may be used to model the effects of the spray root at the boat's leading edge. Consequently, in the present analysis, the drag and the lift due to the pressure on the boat's bottom are no longer related in a simple manner. A further discussion of this point is included in §7.

Inspection of figure 4(*d*) and 4(*e*) shows that, as far as drag and lift are concerned, pressure effects dominate resultant effects at low speeds and vice versa at high speeds. If we identify the effects of forces and moment due to resultants with effects of the spray root, then the discussion would be similar to that of Squire (1957, p. 59 and figure 7).

Figure 5 exhibits the quantities x'_3 and x'_4 which, respectively, locate the boat's leading and trailing edges relative to the boat co-ordinate x'. Examination of figure 5 reveals that x'_4 very rapidly decreases to the value $x'_4 = b$ in (5.2). In addition, x'_3 continues to increase and the length

$$l = x'_4 - x'_3 \tag{5.6}$$

of the wetted surface continues to decrease.

In the course of computation of the example discussed above, it was also observed that for slow speeds ($F < 0{\cdot}1413$) the pressure difference $\hat{p} - p_0$ became negative in the neighbourhood of the leading edge; this negative pressure difference is also mentioned by Saunders (1957, p. 205) but in a different context. From physical considerations, it is not possible for an inviscid fluid in the absence of surface tension to sustain a negative pressure difference $\hat{p} - p_0$. In spite of this, we must remember that the net force acting on the boat is composed of two parts: one due to the pressure distribution $\hat{p} - p_0$ and the other due to the resultants $\mathscr{D}_R$, $\mathscr{L}_R$ at its leading edge. Since it is the negative of this net force which appears in the integral balance law of linear momentum of the fluid sheet and since all balance laws are satisfied exactly by the present solution, it is not clear to what extent we should expect $\hat{p}-p_0$ to be nonnegative near the bow of the boat. It is possible, however, to define an associated nonnegative function $\Delta\hat{p}$ for the pressure difference namely $\Delta\hat{p} = \hat{p}-p_0$ if $\hat{p}-p_0 \geqslant 0$ or $\Delta\hat{p} = 0$ if $\hat{p}-p_0 < 0$, and employ this function in place of $\hat{p}-p_0$ in carrying out the numerical evaluation of the integrals in (4.4e, g, i). If this associated function $\Delta\hat{p}$ is used, then the quantitative results presented in figures 3 and 4 would be altered slightly but qualitatively there would be no change. It should be noted also that, in the calculations for most of the range of Froude number considered (approximately $F \geqslant 0{\cdot}1413$), the pressure difference is never negative and hence $\Delta\hat{p} = \hat{p}_0-p_0$ for $x_3 < x \leqslant x_4$.

Experience suggests that at slow speeds there will be a wave just ahead of the boat whose amplitude decays quite rapidly at distances away from the boat. In the solution presented here (figures 3–5), it is assumed that far ahead of the boat the fluid flows as a uniform stream and this leads to the result that the only possible solution is the uniform stream for the whole region ahead of the boat (recall that in the analysis of the boat problem presented here, only speeds for which $F < 1$ have been considered). Therefore, it is not too surprising that the direct theory of this paper does not completely describe the motion in the neighbourhood of the boat's leading edge. If, on the other hand, it is desired to allow for a wave in the leading region, then more information about the boat's leading edge would be required and the solution would be correspondingly more complex.

In the context of the present formulation of the problem of transition to planing, it should be clear that other hull geometries can be analyzed similarly and without further difficulties. This may be contrasted with the fact that the linear analytical techniques commonly used to discuss the planing of a boat with a simple hull geometry (for references, see Wehausen & Laitone 1960) necessarily become more intricate when more general hull geometries are to be considered.

Since the inviscid theory cannot completely predict the drag force, it was felt desirable to consider the effect of viscous drag and examine to what extent the absence of viscous effects influences the accurate prediction of the boat's orientation. To this end, a frictional drag force was calculated with the use of an empirical formula which utilizes the coefficient of friction C_f given by the I.T.C.C. 1957 model-ship correlation line (see Comstock 1967). Then, the resultant forces and moment applied to the boat were calculated by assuming that the frictional drag force acts along the tangent to the boat's bottom surface. After generalizing the equations of motion (4.10) to include frictional effects, the example of the transition to planing was reconsidered. The predictions of the sinkage and the bow-up trim angle were found to be almost identical to those shown in figures 4(a, b), which confirms the assumption that the orientation of the boat can be accurately predicted by a purely inviscid theory. Supplementary to this, the propulsion force was also calculated in the presence of viscous drag. As to be expected, it was found that the effect of viscosity becomes progressively more important at higher speeds. The result of this calculation for the propulsion force is shown by curve B in figure 4(c).

6. Equations governing the motion of a free-floating wedge

Authors who have previously dealt with the planing problem have often confined attention to plane hull geometries, evidently due to the mathematical simplifications associated with such shapes. Squire (1957), for example, employs the linearized three-dimensional equations of an incompressible inviscid fluid and considers the planing of a wedge over a fluid of infinite depth. In addition, he restricts the motion of the wedge in such a way that the trim angle is constant. Motivated partly by Squire's analysis, we consider in this section the planing of a wedge over a fluid of finite depth but also allow the wedge to float freely. For this purpose, it is necessary to introduce a slight modification of the previous formulation of the planing problem (§§3–4) in order to allow the treatment of a wedge which has a hull geometry with a discontinuous slope at its bottom corner (the previous formulation in §§3–4 dealt with hull geometries which have continuous curvature). Furthermore, we assume that the back surface of the wedge is never vertical.

In his treatment of the planing of a wedge, Squire (1957) assumes that the flow separates from the wedge's bottom corner for all speeds above zero and sets the atmospheric pressure at this corner equal to zero. In the context of the direct approach of this paper, we saw in §3 that the location of the trailing edge is determined by the condition (3.30) which requires the pressure difference $\hat{p}-p_0$ to vanish there. When using the direct approach to study the motion of a wedge, it is preferable to abandon this condition and instead to specify the location of the trailing edge to be the wedge's bottom corner. It then follows that the pressure difference $\hat{p}-p_0$ will not, in general, vanish there.

Although the predictions of a theory which assumes the flow separates from the wedge's bottom corner at all speeds above zero have definite limitations at very slow speeds, we follow Squire and assume that the fluid separates smoothly from the wedge corner. As in §3, this problem can also be analysed by considering separately the three regions shown schematically in figure 1. In regions I and III the equations of motion of the fluid are given by (3.17) and (3.19), respectively, and those in region II reduce to

$$\phi = H_4+K_3(x-x_4), \quad \phi u = k_2, \tag{6.1a, b}$$

$$p = p_0\phi+\rho^*\left[B\phi-\tfrac{1}{2}g\phi^2-\frac{1}{2}\frac{k_2^2}{\phi}(1+\tfrac{1}{3}K_3^2)\right], \tag{6.1c}$$

$$\hat{p} = p_0+\rho^*\left[B-g\phi-\frac{1}{2}\frac{k_2^2}{\phi^2}(1-\tfrac{1}{3}K_3^2)\right], \tag{6.1d}$$

$$\bar{p} = p_0+\rho^*\left[B-\frac{1}{2}\frac{k_2^2}{\phi^2}-\frac{1}{3}\frac{k_2^2K_3^2}{\phi^2}\right], \tag{6.1e}$$

$$C_2 = \tfrac{1}{2}\rho^*\phi u(u^2+\tfrac{1}{3}w^2+g\phi)+pu = \rho^*k_2B+p_0k_2, \tag{6.1f}$$

where $x = x_4$ locates the corner of the wedge, H_4 is the height of this corner and K_3 is the slope of the wedge's planing surface.

In order to match the solutions in the three regions, we again use the jump conditions associated with the integral conservation laws. We introduce the resultants (3.11), as well as the matching conditions (3.12) and (3.14), and also require the fluid to separate smoothly from the trailing edge so that the matching conditions (3.15) and (3.16) still hold. In view of the simplification associated with the planing of a wedge, the results (3.20) now hold and $K_4 = K_3$.

We again assume that far ahead of the wedge the fluid flows as a uniform stream, impose the conditions (3.22) and obtain the results (3.23). Confining attention to the range of speeds for which the Froude number $0 < F < 1$, it follows from the development in §3 that the only solution ahead of the wedge is the uniform stream solution (3.28). We also recall from §§3 and 5 that the solution of (3.19*d*) for the free surface can be obtained in terms of an elliptic integral of the first kind or solved numerically by using (5.5).

Next, we recall from §3 that the location of the leading edge is determined by the condition (3.29). Using (3.20*b*), (3.28) and (3.29) we have

$$\phi|_{x_3=x_3^+} = \phi_3^+ = H_3 = H_1, \tag{6.2}$$

where $x = x_3$ locates the leading edge. Since K_3 is never zero we may use (6.1*a*) in (6.2) to obtain

$$x_3 = x_4 + \frac{H_1 - H_4}{K_3}. \tag{6.3}$$

In order to determine the orientation (relative to the fixed bottom) of the wedge, we turn to a discussion of the equations of motion of a free-floating body. First we note that the formulae (4.1)–(4.6) and (4.11)–(4.13), as well as the equations of motion (4.10), also hold for the wedge. However, due to the simplicity of the hull geometry of the wedge, the integrals defining $\mathscr{D}_P$, $\mathscr{L}_P$ and $\mathscr{M}_P$ in (4.4) can be evaluated analytically. Thus, with the help of (4.4), (7.1) and (7.2), by integration we obtain

$$\mathscr{D}_P = \rho^*\left[B(H_1 - H_4) - \tfrac{1}{2}g(H_1^2 - H_4^2) + \tfrac{1}{2}k^2(1 - \tfrac{1}{3}K_3^2)\left(\frac{1}{H_1} - \frac{1}{H_4}\right)\right], \tag{6.4a}$$

$$\mathscr{L}_P = -\frac{1}{K_3}\mathscr{D}_P, \tag{6.4b}$$

$$\begin{aligned}\mathscr{M}_P = \left(x_{cm} - x_4 + \frac{H_4}{K_3}\right)\mathscr{L}_P - z_{cm}\mathscr{D}_P + \rho^*\left(1 + \frac{1}{K_3^2}\right) \\ \times\left[\tfrac{1}{2}B(H_1^2 - H_4^2) - \tfrac{1}{3}g(H_1^3 - H_4^3) - \tfrac{1}{2}k^2(1 - \tfrac{1}{3}K_3^2)\ln\frac{H_1}{H_4}\right],\end{aligned} \tag{6.4c}$$

where (3.20*a*) and (3.21*a*) have been used. Assuming that the back surface of the wedge is not perpendicular to the wedge's bottom surface, we may specify the hull geometry of the wedge by

$$\eta(x') = \begin{cases}\eta_1 & \text{for} \quad x' \leqslant x_4', \\ \eta_1 + K(x' - x_4) & \text{for} \quad x' > x_4',\end{cases} \tag{6.5}$$

where η_1 and K are constants and $x' = x_4'$ locates the bottom corner of the wedge with respect to the body co-ordinate system x_i' [introduced in §4 following (4.9)]. With this specification, we may use (4.1), (4.2), (4.11), (4.12), (4.13), (6.1*a*) and (6.5) to conclude that

$$x = x'\cos\theta - \eta_1\sin\theta, \quad \phi = z_{cm} + x'\sin\theta + \eta_1\cos\theta \quad (x' \leqslant x_4'), \tag{6.6a, b}$$

as well as

$$K_3 = \tan\theta, \quad x_4 = x_4'\cos\theta - \eta_1\sin\theta, \tag{6.7a, b}$$

$$z_{cm} = H_4 - x_4' \sin\theta - \eta_1 \cos\theta. \tag{6.7c}$$

Once the characteristics of the free-floating wedge are specified in terms of

$$\mathbf{r}_p,\ W,\ \gamma,\ \eta_1,\ K,\ x_4' \tag{6.8}$$

and the parameters (4.15) as well as the dissipation function Φ are known, then (4.4), (4.6), (4.10*a, b*), (4.13), (6.3), (6.4) and (6.7) can be used to solve for the unknowns (4.16).

Before closing this section, we note that Squire (1957) considers the motion of a wedge with a constant trim angle, while in the above development the trim angle is allowed to change. The case of a constant trim angle, however, can be dealt with by simply applying an external trim moment $\mathscr{M}_{\text{ext}} = \mathscr{M}_{\text{ext}}\,\mathbf{e}_2$ which can maintain a constant trim angle throughout the motion. Such an external trim moment is calculated from the angular momentum equation, i.e.

$$\mathscr{M}_{\text{ext}} + \mathscr{M} + \mathbf{r}_p \times \mathbf{F}_p = 0, \tag{6.9}$$

which differs from (4.10*b*) due to the presence of the external trim moment. Under these circumstances, θ (defined in (4.11)) is specified and the procedure for obtaining the solutions for the unknowns

$$z_{cm},\ x_3,\ x_4,\ F_p,\ \mathscr{M}_{\text{ext}} \tag{6.10}$$

is the same as that discussed above except that (4.10*b*) is replaced by (6.9), from which $\mathscr{M}_{\text{ext}}$ can be calculated.

Now in line with (4.5), let the net resultant force $\mathscr{F}_R$, applied by the fluid on the wedge, be expressed in the form

$$\mathscr{F}_R = \mathscr{D}_R \mathbf{e}_1 + \mathscr{L}_R \mathbf{e}_3. \tag{6.11}$$

Then, on substitution of (3.20) and (4.6) into (6.10), we obtain

$$\mathscr{F}_R = \left[-\tfrac{1}{6}\rho^* \frac{k^2}{H_3}(K_2^3 - K_3^3) + \frac{\Phi H_3}{k}\right]\mathbf{e}_1 - \tfrac{1}{3}\rho^* \frac{k^2}{H_3}(K_3 - K_2)\mathbf{e}_3. \tag{6.12}$$

From the expression (4.1) for the position vector $\hat{\mathbf{r}}$ of the top surface of the fluid, the tangent vector to the wedge's bottom surface is $\mathbf{r}_x = \mathbf{e}_1 + K_3 \mathbf{e}_3$ (since ϕ is given by (6.1*a*)); and it is then clear that, in general, $\mathscr{F}_R$ is not perpendicular to the wedge's bottom surface. But, this is only one part of the net force $\mathscr{F}$ in (4.5*a*). The remaining part, which is due to the integrated effect of the pressure, is always normal to the wedge's bottom surface. It seems reasonable to expect that, if the problem were formulated with the use of the nonlinear three-dimensional theory, no discontinuities would be present and the net force would act normal to the wedge's bottom surface. We observe that, in the context of the theory of fluid sheets employed here, it is possible to require $\mathscr{F}_R$ in (6.12) to act normal to the wedge's bottom surface by specifying the rate of energy dissipation Φ to be of the form

$$\Phi = \tfrac{1}{6}\rho^* \frac{k^3}{H_3^2}(K_2 - K_3)^2. \tag{6.13}$$

Under these circumstances the drag $\mathscr{D}_T$ and lift $\mathscr{L}_T$ are simply related by the formula

$$\mathscr{D}_T = -K_3 \mathscr{L}_T \tag{6.14}$$

and it follows that the net force $\mathscr{F}$ acts normal to the wedge's bottom surface.

7. Two examples of free-floating bodies: a wedge-like boat and a wedge

In line with the remarks made in § 6, we consider here two examples concerning the transition to planing of a wedge-like boat and a wedge; a wedge-like boat is one whose hull geometry has continuous curvature and yet is wedge-like in shape. Our aim is to examine to what extent the more simplified formulation of the wedge can be used to predict the main features of the planing problem.

It was assumed in § 6 that the fluid separates smoothly from the wedge's bottom corner for all speeds above zero. Under this condition, as noted earlier [see remarks preceding (6.1)], the resulting predictions have definite limitations at very slow speeds. In order to estimate the quantitative nature of these limitations, we compare the predictions of the theory of §§ 3 and 4 for a wedge-like boat with those of § 6 for a wedge.

First, we consider a wedge-like boat by specifying its equilibrium depth H_1 and the function η in (4.12) to be of the form (5.1) with

$$\left.\begin{aligned} \eta_1 &= -0{\cdot}5\,\mathrm{m}, \quad A = 0\,\mathrm{m}^{-1}, \quad B = 10\,\mathrm{m}^{-1}, \\ a &= -8{\cdot}8\,\mathrm{m}, \quad b = 1{\cdot}2\,\mathrm{m}, \quad H_1 = 10\,\mathrm{m}, \end{aligned}\right\} \tag{7.1}$$

where the quantities W, γ, x'_p, z'_p and the dissipation function Φ are given by (5.3). Again we use the same computer program as that discussed in § 5 to solve for the motion of the wedge-like boat. The curves labelled A in figure 6(a, b, c) represent, respectively, the resulting solution for the quantities z_{cm}, θ and F_p/W.

Next, we use the theory of § 6 to consider the motion of a free-floating wedge specified by (6.5) and, in order for this wedge to correspond to the wedge-like boat as closely as possible, we take

$$\eta_1 = -0{\cdot}5\,\mathrm{m}, \quad K = 8{\cdot}144\,\mathrm{m}, \quad x'_4 = 1{\cdot}2\,\mathrm{m}, \quad \mathbf{x}_1 = 10\,\mathrm{m} \tag{7.2}$$

and again suppose that quantities W, γ, x'_p, z'_p and Φ are given by (5.3). Here it should be noted that, in assuming that the fluid separates from the wedge's bottom corner, the numerical value of K in (6.5) does not influence the solution for the motion of a wedge. The equations (4.10), (6.4) and (6.7) can then be used to solve by iteration for the unknowns (4.6) with the aid of a computer program which is even simpler than that used in § 5. The curves labelled B in figure 6(a, b, c) represent, respectively, the resulting solution for the quantities z_{cm}, θ and F_p/W.

An examination of figure 6(a, b, c) reveals that the quantitative differences between the curves A and B are relatively small except near zero speed and near the hump speed. Moreover, these results appear to be qualitatively the same as those obtained in § 5.

We now briefly consider to what extent the direct approach of this paper can be used to formulate the wedge problem for a range of speeds which includes the zero speed. This can be done by introducing resultants analogous to those in (3.11) but acting at the corner of the wedge and by exploiting a condition similar to (3.30). For this purpose, let the wedge's back surface be specified by

$$\phi = H_4 + K_5(x - x_4), \tag{7.3}$$

where K_5 is a constant and let $x = x_5$ locate the trailing edge on this surface. Taking

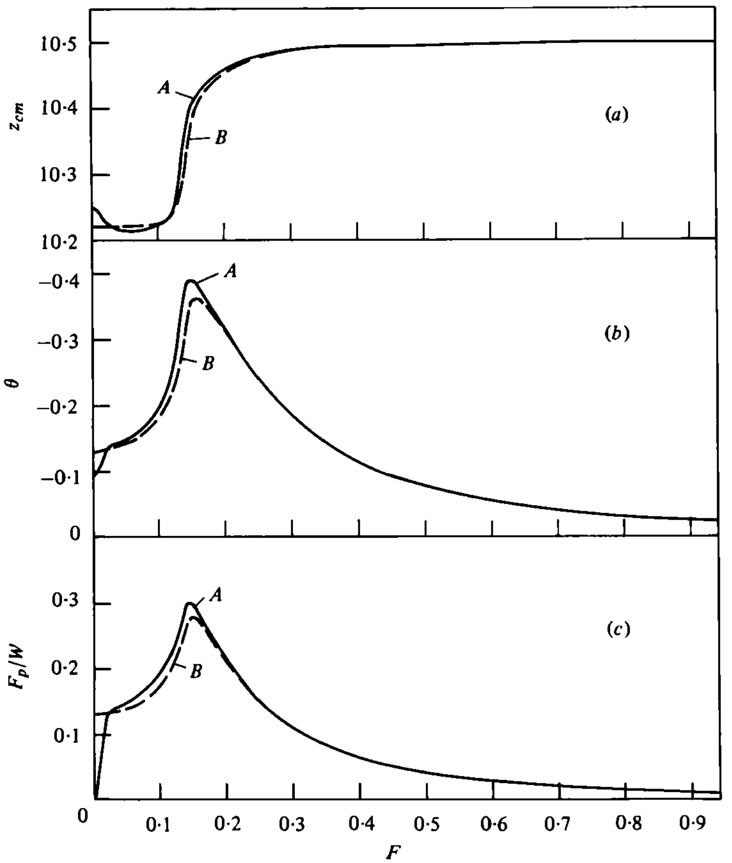

FIGURE 6. Transition to planing of a wedge and a wedge-like boat, some aspects of the solutions represented by dashed curves (wedge) and solid curves (wedge-like boat): (*a*) the height of the centre of mass z_{cm} in metres; (*b*) the value of θ in radians; and (*c*) the normalized propulsion force F_p/W, where the weight of the boat $W = 7000\ \mathrm{N\ m^{-1}}$.

$\Phi = 0$, we may use (3.30) to determine the height of the trailing edge $\phi|_{x=x_s} = H_5$ by the condition

$$gH_1 + \frac{1}{2}\frac{k^2}{H_1^2} + \frac{1}{2}\frac{k^2}{H_5^2}(\tfrac{1}{3}K_5^2 - 1) - gH_5 = 0, \tag{7.4}$$

where (3.20*f*), (3.21*a*), (3.23*c*) and (7.3) have also been used.

When the speed $u_1 = 0$, $k = 0$ and the solution of (7.4) is $H_5 = H_1$. Now for many wedge shapes of interest the slope of the back surface is so large that

$$\tfrac{1}{3}K_5^2 - 1 > 0. \tag{7.5}$$

It therefore follows that there exists a finite speed range (which excludes the value zero but includes speeds which are arbitrarily close to zero) for which

$$H_5 > H_1. \tag{7.6}$$

Since physically we expect $H_5 \leqslant H_1$, this alternative formulation of the wedge problem should be discarded. It is of interest, however, to note that this deficiency is caused by the fact that the idealization of a wedge-shaped boat requires the curvature to be zero on its back surface.

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Appendix

We include here a brief development of the jump conditions utilized in the formulation of the flow past an obstacle given in §§3 and 4. These jump conditions, which are associated with the restricted theory of a directed fluid sheet (Green & Naghdi 1977), are applicable to the steady two-dimensional motion of an incompressible homogeneous inviscid fluid over a level bottom. We recall from §2 that, for two-dimensional motion in the x, z plane, we may identify y with θ^2 and set $x = \chi(\theta^1, t)$. Then, it follows from the form of the position vector $\mathbf{r}$ in $(2.3a)$ that the base vectors $\mathbf{a}_i$ $(i = 1, 2, 3)$ and their reciprocals $\mathbf{a}^i$ are given by

$$\left.\begin{aligned} \mathbf{a}_1 = (\mathbf{e}_1 + \psi_x \mathbf{e}_3)\frac{\partial \chi}{\partial \theta^1}, \quad \mathbf{a}_2 = \mathbf{e}_2, \quad a^{\frac{1}{2}}\mathbf{a}_3 = (-\psi_x \mathbf{e}_1 + \mathbf{e}_3)\frac{\partial \chi}{\partial \theta^1}, \\ \mathbf{a}^1 = a^{-1}\mathbf{a}_1, \quad \mathbf{a}^2 = \mathbf{a}_2, \quad \mathbf{a}^3 = \mathbf{a}_3, \end{aligned}\right\} \tag{A1}$$

where the function ψ and the basis $\mathbf{e}_i$ are defined in §2 and

$$a = (1 + \psi_x^2)\left(\frac{\partial \chi}{\partial \theta^1}\right)^2. \tag{A2}$$

We also recall that the quantities $\mathbf{S}^\alpha$ and $\mathbf{s}$ defined in Green & Naghdi (1977) in the case of two-dimensional motions can, without loss in generality, be specified by

$$a^{\frac{1}{2}}\mathbf{S}^1 = S\mathbf{e}_2, \quad \mathbf{S}^2 = 0, \quad (1 + \psi_x^2)^{\frac{1}{2}}\mathbf{s} = s\mathbf{e}_2, \tag{A3}$$

where S and s which occur in $(2.7a, b)$ are arbitrary functions of x and t. Consider a singular curve (or a curve of discontinuity) on the surface $\mathscr{s}$ of the directed fluid sheet $\mathscr{C}$. For the present purpose, it will suffice to assume the singular curve to be stationary and identify its fixed location by the line $x = x_3$ (corresponding to the leading edge of the body in §3). Then, with the help of (A3), as well as the balance laws of the theory of a directed fluid sheet (see, for example, (6.4) of Naghdi 1979), the jump conditions for two-dimensional motions over a level bottom are†

$$\left.\begin{aligned} [\![\rho^*\phi u]\!]_3 &= 0, \\ [\![\rho^*\phi u(u\mathbf{e}_1 + \tfrac{1}{2}w\mathbf{e}_3) + p\mathbf{e}_1]\!]_3 &= \mathbf{F}, \\ [\![\tfrac{1}{12}\rho^*\phi uw\mathbf{e}_3 + S\mathbf{e}_1]\!]_3 &= \mathbf{L}, \\ [\![\tfrac{1}{2}\rho^*\phi u(2\epsilon + u^2 + \tfrac{1}{3}w^2 + g\phi) + pu + a^{\frac{1}{2}}q^1]\!]_3 &= -\Phi + \mathbf{F}\cdot\mathbf{V} + \mathbf{L}\cdot\mathbf{W}, \end{aligned}\right\} \tag{A4}$$

where the notation $[\![f]\!]_3$ for the jump in f across $x = x_3$ is defined by (3.13), the relations $(2.13a, b)$ have been employed,

† The balance laws used here correspond to those used by Naghdi (1979) if we put $\bar{k} = 0$, $k = 1/12$, $\Omega = g\psi$.

$$\left.\begin{aligned}\mathbf{F} &= \lim_{\delta\to 0}\int_{x_3-\delta}^{x_3+\delta}[\hat{p}\phi_x\,\mathbf{e}_1+(\overline{p}-\hat{p})\,\mathbf{e}_3]dx, \quad \mathbf{L} = \lim_{\delta\to 0}\int_{x_3-\delta}^{x_3+\delta}[\tfrac{1}{2}\hat{p}\phi_x\,\mathbf{e}_1-\tfrac{1}{6}(\hat{p}+\overline{p})\,\mathbf{e}_3]dx,\\ \mathbf{F}\,.\,\mathbf{V} &= \lim_{\delta\to 0}\int_{x_3-\delta}^{x_3+\delta}[\hat{p}\phi_x u+\tfrac{1}{2}(\overline{p}-\hat{p})w]dx, \quad \mathbf{L}\,.\,\mathbf{W} = \lim_{\delta\to 0}\int_{x_3-\delta}^{x_3+\delta}[-\tfrac{1}{6}(\hat{p}+\overline{p})w]dx,\\ \Phi &= -\lim_{\delta\to 0}\int_{x_3-\delta}^{x_3+\delta}\rho^*\phi r_c\,dx\end{aligned}\right\} \quad (\text{A }5)$$

and where the terms ϵ, q^1 and r_c represent, respectively, the internal energy per unit mass, the component of the heat flux vector per unit length along $\mathbf{a}_1$ and the rate of heat supply per unit mass due to the fluxes through the top and bottom surfaces of the fluid sheet. The conditions (A 4) represent the jumps associated with the balance of mass, linear momentum, director momentum and energy, respectively. Since in the present paper the position vector $\mathbf{r}$ and director $\mathbf{d}$ are assumed to be continuous across the singular curve, the jump conditions associated with angular momentum is identically satisfied. Furthermore, we note that in defining the resultants $\mathbf{F}$, $\mathbf{L}$, $\mathbf{F}\,.\,\mathbf{V}$, $\mathbf{L}\,.\,\mathbf{W}$ and Φ in (A 5) it has been assumed that the pressures $\hat{p}$ and $\overline{p}$, as well as the rate of heat supply r_c may be unbounded on the singular line.

It is now convenient to recall from the appendix of Green & Naghdi (1976*a*) the results that in regions free of discontinuities and for an inviscid incompressible fluid at constant temperature:

$$\epsilon = \text{constant}, \quad q^1 = 0, \quad r_c = 0, \qquad (\text{A }6)$$

where we have used the fact that r_c arises from contributions of the heat flux through the top and bottom surfaces of the fluid sheet. Although (A 6) holds in regions free of discontinuities, it is still possible for r_c to be unbounded on $x = x_3$ and for the resultant Φ in (A 5) to be non-zero there.

Next, we combine the expressions $\mathbf{F}\,.\,\mathbf{V}$ and $\mathbf{L}\,.\,\mathbf{W}$ in (A 5) and use (2.13*a*, *b*) to obtain the rate of work expression

$$\mathbf{F}\,.\,\mathbf{V}+\mathbf{L}\,.\,\mathbf{W} = \lim_{\delta\to 0}\int_{x_3-\delta}^{x_3+\delta} -\hat{p}\,\frac{\partial\phi}{\partial t}dx. \qquad (\text{A }7)$$

Clearly, if the motion is steady, then $\partial\phi/\partial t = 0$ and we conclude that

$$\mathbf{F}\,.\,\mathbf{V}+\mathbf{L}\,.\,\mathbf{W} = 0. \qquad (\text{A }8)$$

It then follows from the first two of (A 6) and (A 8) that the component forms of the jump conditions (A 4) are given by

$$\left.\begin{aligned}&[\![\rho^*\phi u]\!]_3 = 0, \quad [\![\rho^*\phi u^2+p]\!]_3 = F_1,\\ &[\![\tfrac{1}{2}\rho^*\phi uw]\!]_3 = F_3, \quad [\![\tfrac{1}{12}\rho^*\phi uw]\!]_3 = L_3,\\ &[\![\tfrac{1}{2}\rho^*\phi u\,(u^2+\tfrac{1}{3}w^2+g\phi)+pu]\!]_3 = -\Phi,\end{aligned}\right\} \qquad (\text{A }9)$$

and

$$[\![S]\!]_3 = L_1, \qquad (\text{A }10)$$

where F_1, F_3, L_1, L_3 are the nonzero components of $\mathbf{F}$, $\mathbf{L}$ referred to the basis $\mathbf{e}_i$, i.e.

$$\mathbf{F} = F_1\mathbf{e}_1+F_3\mathbf{e}_3, \quad \mathbf{L} = L_1\mathbf{e}_1+L_3\mathbf{e}_3. \qquad (\text{A }11)$$

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